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THE RESONANT FREQUENCY OF A RECIPROCATING LOAD ON AN ELASTIC BEAM

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The resonant frequency of a reciprocating point load on an elastic beam is considered. Employing the elementary beam theory and applying the Fourier expansion technique, the exact solution is obtained in the form of double series. We have found that the resonant frequency of the reciprocating load is given by $\omega^*_{n,m} = \omega^*_n/m$, where ω^*_n is the natural frequency of the nth mode of the beam vibration and m is an integer ranging from 1 to infinity. Checking the vibration mode, we have concluded that there is an infinite number of resonant frequencies for a single mode of vibration. This multi-resonance for a single mode may prove to be quite interesting to engineers.

1. INTRODUCTION

The dynamic response of an elastic beam to a moving load is one of the classical and basic engineering problems. A unified treatment for a uniformly moving load on the beam can be found in the book by Fryba [1]; however, there are fewer studies for a reciprocating load. We can cite the works of Ohyoshi [2, 3], Goloskokov [4] and Watanabe [5–7] regarding the reciprocating load.

Ohyoshi [3] has considered the dynamic response of an elastic half-space to a reciprocating load, including the frictional effect, and has given more accurate information for the internal stress field, which might be more helpful for tribologists. Goloskokov [4] has considered the time harmonic response of a cylindrical shell. Watanabe [7] has considered the reciprocating and vibrating anti-plane load on an elastic half-space, as a model of a imperfect positioning of a pin load, and has derived an approximate expression for the response. All of these authors have considered only the dynamic response, not the resonance characteristics. Here, we have considered the resonant frequency for the reciprocating load and have found an infinite number of resonant frequencies for a single mode of vibration. This characteristic of the multi-resonant frequencies is very strange and interesting.

2. STATEMENT OF THE PROBLEM

Let us consider an elastic beam and take the co-ordinates as shown in Figure 1. A reciprocating point load with an interval, $-a \leqslant x \leqslant +a$, is assumed, as follows:

$$P(x, t) = P_0\delta(x - a\sin(\omega t)), \tag{1}$$

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where P_0 is the magnitude of the load, ω is the frequency of the reciprocation and $\delta(\cdot)$ is the Dirac delta function. Employing elementary beam theory [8, p. 324], we have the simple equation of motion as

$$EI\frac{\partial^4 W}{\partial x^4} + \rho A\frac{\partial^2 W}{\partial t^2} = P_0\delta(x - a\sin(\omega t)), \tag{2}$$

where EI, ρ and A are the bending rigidity, density and cross-sectional area respectively.

In the present paper, the three types of boundary conditions as shown in Figure 1 are considered. They are:

(a) both ends simply supported;

$$W|_{x=\pm l} = 0, \qquad \left.\frac{\partial W^2}{\partial x^2}\right|_{x=\pm l} = 0, \tag{3}$$

(b) both ends fixed,

$$W|_{x=\pm l} = 0, \qquad \left.\frac{\partial W}{\partial x}\right|_{x=\pm l} = 0, \tag{4}$$

(c) one end simply supported and the other fixed,

$$W|_{x=\pm l} = 0, \qquad \left.\frac{\partial W}{\partial x}\right|_{x=-l} = 0, \qquad \left.\frac{\partial W^2}{\partial x^2}\right|_{x=+l} = 0. \tag{5}$$

Expanding the reciprocating load given by the delta function in the form of double Fourier series (see Appendix A), we can obtain the formal solution which is composed of two parts:

$$W(x, t) = W_s(x) + W_d(x, t), \tag{6}$$

where the first and second terms in Equation (6) correspond to the static and dynamic responses respectively, and are given by

$$W_s(x) = \frac{P_0}{2lEI}\left[2\sum_{k=1}^{\infty}\left(\frac{l}{k\pi}\right)^4 \mathrm{J}_0\left(\frac{k\pi a}{l}\right)\cos\left(\frac{k\pi x}{l}\right) + \frac{x^4}{4!} + A_s x + B_s x + C_s x + D_s\right], \tag{7}$$

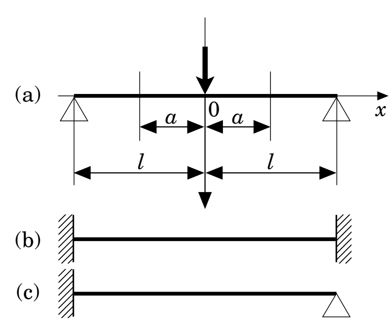

Figure 1. A reciprocating load on an elastic beam and three types of boundary conditions.

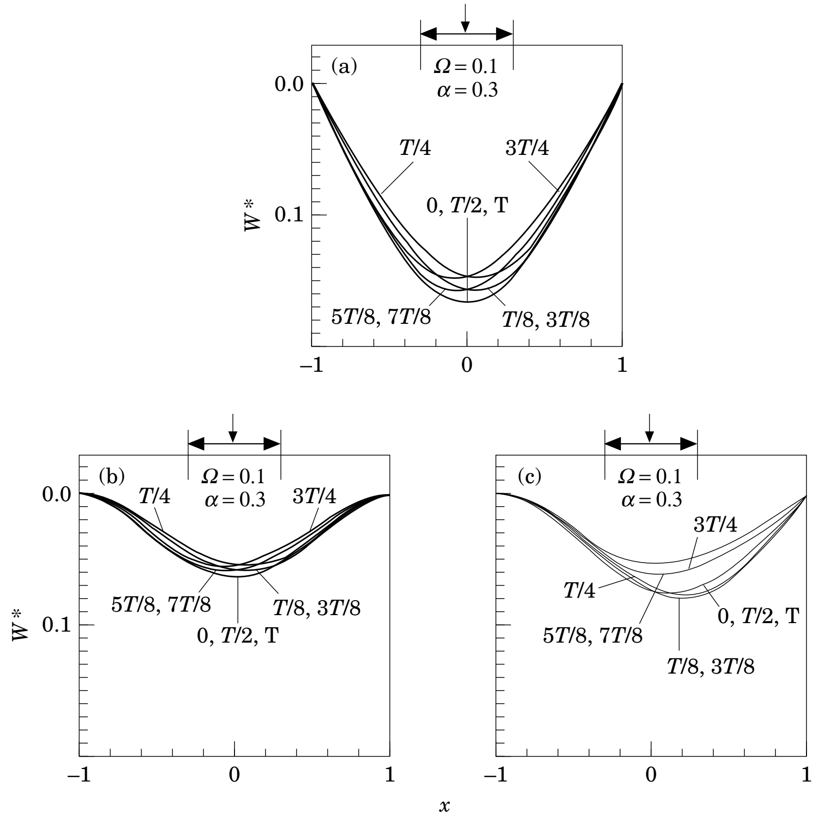

Figure 2. The deflection of beam within a period. (a) Both ends simply supported; (b) Both ends fixed; (c) One end simply supported and the other fixed.

$$W_d(x, t) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \exp(-\mathrm{i}m\omega t) \sum_{k=-\infty}^{+\infty} \Bigg[\frac{P_0}{2lEI} \frac{\mathrm{J}_m(k\pi a/l)}{(k\pi/l)^4 - \beta_m^4} \exp(\mathrm{i}k\pi x/l)$$

$$+ A_d(m, k) \sinh(\beta_m x) + B_d(m, k) \cosh(\beta_m x)$$

$$+ C_d(m, k) \sin(\beta_m x) + D_d(m, k) \cos(\beta_m x) \Bigg], \quad (8)$$

where A_j, B_j, C_j and D_j, $j = s, d$, are unknown coefficients to be determined by the boundary conditions and

$$\beta_m = (\sqrt{|m|\omega})/\lambda. \quad (9)$$

After applying the boundary conditions of equations (3)–(5), we have the exact solution as in Appendix B. Typical deflections of the beam are shown in Figure 2, where the following non-dimensionalization is introduced:

$$X = x/l, \qquad T = 2\pi/\Omega, \qquad \alpha = a/l, \qquad \Omega = \omega l^2 \sqrt{\rho A/EI}, \qquad W^* = (EI/P_0 l^3)W. \quad (10)$$

3. THE RESONANT FREQUENCY

From the exact solution in Appendix B, we can discuss the resonant frequency. For the case (a), one expects that the solution will be divergent when

$$k\pi - \beta_m l = 0, \qquad \cos(\beta_m l) = 0. \tag{11, 12}$$

For the first condition of equation (11), replacing $\beta_m l$ with $k\pi$ in equation (B2), we find that the solution is divergent only when the integer m is an odd number. Thus the resonant frequency is given by

$$\omega = \frac{1}{m}\left(\frac{n\pi}{2l}\right)^2 \sqrt{\frac{EI}{\rho A}}, \tag{13}$$

where $n = 2k$.

On the other hand, for the second case of equation (11), if we substitute $\beta_m l = n\pi/2$ (n is odd) into equation (B2), the solution is divergent only when m is an even number. Then, we have the same expression as equation (12) for the resonant frequency. Therefore, introducing the natural frequency of a simply supported beam,

$$\omega_n^* = \left(\frac{n\pi}{2l}\right)^2 \sqrt{\frac{EI}{\rho A}}, \tag{14}$$

we have the resonant frequency for the reciprocating load as

$$\omega_{n,m}^* = \omega_n^*/m, \qquad m = 1, 2, 3, \ldots. \tag{15}$$

In order to check the vibration mode at resonance, we substitute equation (15) into equation (B2) and then have

$$W_d(x, t) \approx \frac{P_0 l^3}{2EI} \lim_{\omega \to \omega_{n,m}^*} \left[\frac{\mathrm{J}_m(n\pi a/2l)}{(n\pi/2)^4(\omega - \omega_{n,m}^*)}\right] \sin(\omega_n^* t) \sin(n\pi x/2l) \tag{16}$$

for even n and odd m, and

$$W_d(x, t) \approx \frac{P_0 l^3}{2EI} \frac{2\omega_n^*}{(n\pi/2)} \lim_{\omega \to \omega_{n,m}^*} \left[\frac{1}{\omega - \omega_{n,m}^*} \sum_{k = odd}^{\infty} \frac{\mathrm{J}_m(k\pi a/2l)}{(n\pi/2)^2 - (k\pi)^2}\right] \cos(\omega_n^* t) \cos\left(\frac{n\pi x}{2l}\right) \tag{17}$$

for odd n and even m.

Thus, at the resonant frequency $\omega_{n,m}^*$, the frequency of the beam vibration is equal to that of the nth natural frequency, ω_n^*, and the anti-symmetric and symmetric modes of vibration correspond to the even and odd numbers of n respectively. At a resonant with symmetric mode, that is, odd n, the largest resonant frequency $\omega_{n,m}^*$ is half of the resonant frequency of the free vibration, ω_n^*. The reason is that m is even and starts from 2. In Figure 3(a) one can find the contradiction in order between two resonant frequencies marked by ($n = 5$, $m = 2$) and ($n = 4$, $m = 1$). This is very strange because, in the general knowledge of the vibration, the resonant frequency of the lower mode is higher than that of the higher mode.

For the other two cases of the boundary conditions, it is easily checked that the first condition, $\beta_m l = n\pi$, in equation (11) does not give the resonant frequency. After checking the denominator

$$\sin(\beta_m l)\cosh(\beta_m l) \pm \cos(\beta_m l)\sinh(\beta_m l) = 0 \tag{18}$$

in equation (B4) for case (b), we can find the same expression for the resonant frequency as in case (a). In equation (13) the natural frequency is replaced by

$$\omega_n^* = \left(\frac{p_n}{2l}\right)^2 \sqrt{\frac{EI}{\rho A}}, \tag{19}$$

where p_n is the well-known nth root of the eigenequation, $\cos(p)\cosh(p) = 1$. In this case, it is also valid that the denominator m in equation (15) must be even for an odd mode number n and it must also be odd for even n.

For the last case (c), the resonant frequency has the same expression as in the former two cases. However, a replacement for the eigenvalue, p_n, which is the root of the eigenequation

$$\sin(p)\cosh(p) - \cos(p)\sinh(p) = 0, \tag{20}$$

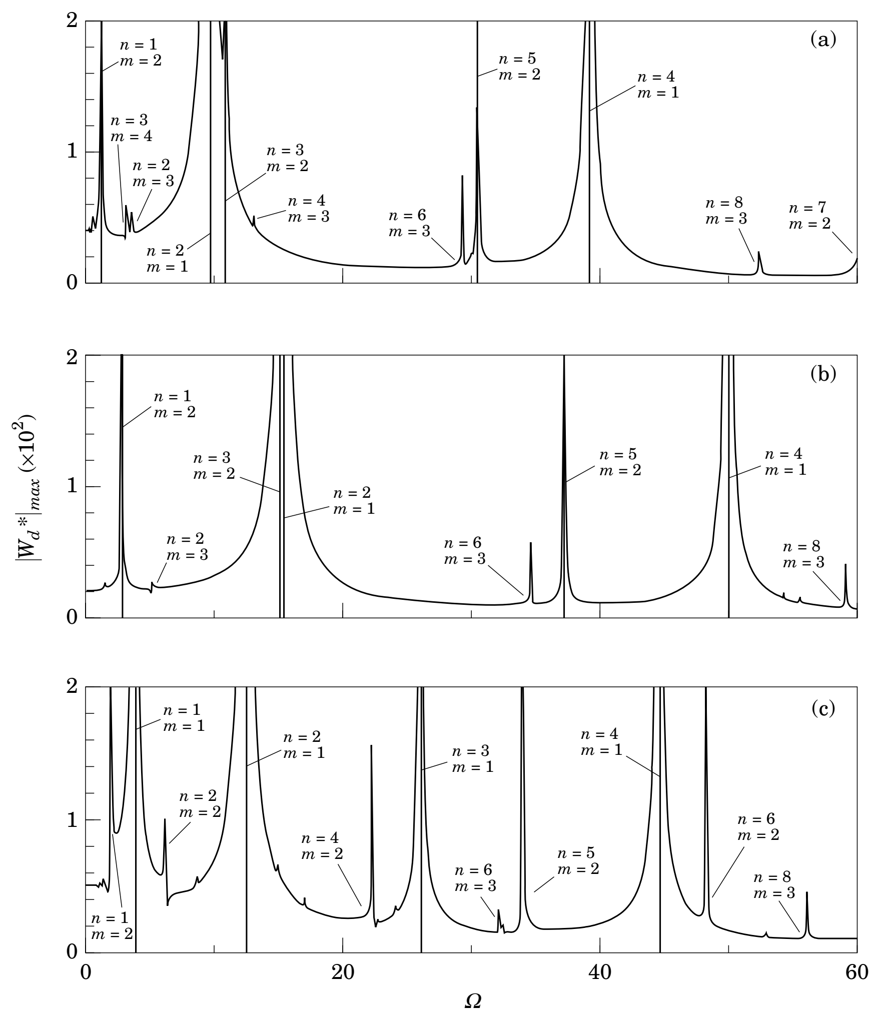

Figure 3. Resonance curves (amplitude versus frequency). (a) Both ends simply supported; (b) Both ends fixed; (c) One end simply supported and the other fixed.

TABLE 1

The resonant frequency $\omega^*_{n,m} = (1/m)(p_n/2l)^2\sqrt{EI/\rho A}$ *of the reciprocating load on an elastic beam*

Boundary conditions	Eigenvalues	Conditions for m and n
(a) Both ends simply supported	$p_n = n\pi$	(1) If n is odd, m is even (2) If n is even, m is odd
(b) Both ends fixed	$\cos(p_n)\cosh(p_n) = 1$	(3) If n is odd, m is even (4) If n is even, m is odd
(c) one end simply supported and the other fixed	$\tan(p_n) = \tanh(p_n)$	None for all m and n

should be used, and the contradiction in order between the mode number and its highest ($m = 1$) resonant frequency does not take place, as there is no restriction for the combination of even and odd numbers of m and n.

Consequently, for all three cases, the resonant frequency $\omega^*_{n,m}$ of the reciprocating load is given by equation (14), in which ω^*_n is the natural frequency of the beam corresponding to the boundary condition. As the denominator m ranges from 1 to infinity, there is an infinite number of resonant frequencies for a single mode of vibration. This may be called *multi-resonance*. The resonant curves (amplitude–frequency diagrams are shown in Figure 3). It is evident that the lower resonant frequency, with a larger value of m, contributes less to the amplitude. For example, if we focus our attention on the fourth mode, the first resonance with $m = 1$ is very strong; however, the second with $m = 3$ in Figure 3(a) is weak and, especially in Figure 3(b), the second is invisible. In a practical problem, this multi-resonance effect may be weak because every kind of material has internal friction.

4. CONCLUSIONS

The resonant frequency of the reciprocating load on an elastic beam has been discussed. We have found that the resonant frequency is $1/m$ times lower than the natural frequency of the beam with the corresponding edge condition and that there is an infinite number of resonant frequencies for a single mode of vibration as the integer m ranges from 1 to infinity. This is multi-resonance. Our results are summarized in Table 1.

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APPENDIX A: FOURIER EXPANSION OF THE RECIPROCATING LOAD IN EQUATION (2)

Let us consider the Fourier expansion of the reciprocating load given by the delta function which is on the right side of equation (2); that is,

$$P_0\delta(x - a\sin(\omega t)) = P_0 \sum_{k=-\infty}^{+\infty} a_k(t)\exp(\mathrm{i}k\pi x/l), \qquad -l < x < +l, \tag{A1}$$

where $a_k(t)$ is the Fourier coefficient. In order to determine the Fourier coefficient $a_k(t)$, we multiply both sides of equation (A1) by $\exp(-\mathrm{i}k\pi x/l)$ and apply the integration formula,

$$g(c) = \int_a^b g(x)\delta(x-c)\,\mathrm{d}x, \qquad a < c < b, \tag{A2}$$

where $g(\cdot)$ is an arbitrary function. Then we have

$$P_0\delta(x - a\sin(\omega t)) = \frac{P_0}{2l}\sum_{k=-\infty}^{+\infty} \exp\left\{-\frac{\mathrm{i}k\pi a}{l}\sin(\omega t)\right\}\exp(\mathrm{i}k\pi x/l). \tag{A3}$$

The exponential in the summation of the above equation (A3) is just the form of the generating function of the Bessel function. Thus, using Jacobi's expansion formula [9, p. 22],

$$\exp(-\mathrm{i}z\sin\theta) = \sum_{m=-\infty}^{+\infty} \mathrm{J}_m(z)\exp(-\mathrm{i}m\theta), \tag{A4}$$

we have the double Fourier series for the reciprocating load,

$$P_0\delta(x - a\sin(\omega t)) = (P_0/2l)\sum_{m=-\infty}^{+\infty}\exp(-\mathrm{i}m\omega t)\sum_{k=-\infty}^{+\infty}\mathrm{J}_m(k\pi a/l)\exp(\mathrm{i}k\pi x/l). \tag{A5}$$

Substituting equation (A5) into equation (2), we can seek the particular solution in the form of double series as

$$W(x,t) = \sum_{m=-\infty}^{+\infty}\exp(-\mathrm{i}m\omega t)\sum_{k=-\infty}^{+\infty} W_{k,m}\exp(\mathrm{i}k\pi x/l), \tag{A6}$$

where the $W_{k,m}$ are unknowns to be determined. The final form of the general solution is given by equations (6)–(8).

APPENDIX B: EXACT SOLUTIONS

(a) Both ends simply supported:

$$W_s(x) = \frac{P_0 l^3}{2EI}\left\langle \frac{1}{24}\{(x/l)^2 - 5\}\{(x/l)^2 - 1\} + \sum_{k=1}^{\infty} \frac{\mathrm{J}_0(k\pi a/l)}{(k\pi)^4}\left[2\cos\left(\frac{k\pi x}{l}\right) + (-1)^k[(k\pi)^2\{(x/l)^2 - 1\} - 2]\right]\right\rangle, \quad \text{(B1)}$$

$$W_d(x, t) = \frac{P_0 l^3}{EI}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mathrm{J}_m(k\pi a/l)}{(k\pi)^4 - (\beta_m l)^4}\left\langle \cos\left(m\omega t - \frac{k\pi x}{l}\right) + (-1)^m \cos\left(m\omega t + \frac{k\pi x}{l}\right) - \frac{(-1)^k\{1 + (-1)^m\}}{2(\beta_m l)^2}\cos(m\omega t)\left[\{(\beta_m l)^2 - (k\pi)^2\}\frac{\cosh(\beta_m x)}{\cosh(\beta_m l)} + \{(\beta_m l)^2 + (k\pi)^2\}\frac{\cos(\beta_m x)}{\cos(\beta_m l)}\right]\right\rangle. \quad \text{(B2)}$$

(b) Both ends fixed:

$$W_s(x) = \frac{P_0 l^3}{2EI}\left[2\sum_{k=1}^{\infty}\frac{\mathrm{J}_0(k\pi a/l)}{(k\pi)^4}\{\cos(kx/l) - (-1)^k\} + \tfrac{1}{12}(x/l + 1)^2(x/l - 1)^2\right], \quad \text{(B3)}$$

$$W_d(x, t) = \frac{P_0 l^3}{EI}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mathrm{J}_m(k\pi a/l)}{(k\pi)^4 - (\beta_m l)^4}\left\langle \cos\left(m\omega t - \frac{k\pi x}{l}\right) + (-1)^m \cos\left(m\omega t + \frac{k\pi x}{l}\right) - (-1)^k\left[\{1 - (-1)^m\} \times \frac{k\pi}{\beta_m l}\frac{\sin(\beta_m l)\sinh(\beta_m x) - \sinh(\beta_m l)\sin(\beta_m x)}{\sin(\beta_m l)\cosh(\beta_m l) - \cos(\beta_m l)\sinh(\beta_m l)}\sin(m\omega t) + \{1 + (-1)^m\} \times \frac{\sin(\beta_m l)\cosh(\beta_m x) + \sinh(\beta_m l)\cos(\beta_m x)}{\sin(\beta_m l)\cosh(\beta_m l) + \cos(\beta_m l)\sinh(\beta_m l)}\cos(m\omega t)\right]\right\rangle. \quad \text{(B4)}$$

(c) One end simply supported and the other fixed:

$$W_s(x) = \frac{P_0 l^3}{2EI}\left\langle \sum_{k=1}^{\infty} \frac{J_0(k\pi a/l)}{(k\pi)^4}\left[2\cos\left(\frac{k\pi a}{l}\right) + (-1)^k\left\{\left(\frac{k\pi}{2}\right)^2 (x/l+1)^2(x/l-1) - 2\right\}\right]\right.$$

$$\left. + \tfrac{1}{24}(x/l+1)^2(x/l-1)(x/l-2)\right\rangle \tag{B5}$$

$$W_d(x,t)$$

$$= \frac{P_0 l^3}{EI}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{J_m(k\pi a/l)}{(k\pi)^4 - (\beta_m l)^4}\left[\cos\left(m\omega t - \frac{k\pi x}{l}\right) + (-1)^m \cos\left(m\omega t + \frac{k\pi x}{l}\right)\right.$$

$$\left. - (-1)^k \frac{\{1+(-1)^m\}W_{m,k}^{(1)}(x)\cos(m\omega t) + (2k\pi/\beta_m l)\{1-(-1)^m\}W_m^{(2)}(x)\sin(m\omega t)}{\sin(2\beta_m l)\cosh(2\beta_m l) - \cos(2\beta_m l)\sinh(2\beta_m l)}\right]; \tag{B6}$$

where

$$W_{m,k}^{(1)}(x) = \left[\left\{\left(\frac{k\pi}{\beta_m l}\right)^2 + 1\right\}\sin(\beta_m l)\cosh(\beta_m l) + \left\{\left(\frac{k\pi}{\beta_m l}\right)^2 - 1\right\}\cos(\beta_m l)\sinh(\beta_m l)\right]$$

$$\times \{\sin(\beta_m l)\sinh(\beta_m x) - \sinh(\beta_m l)\sin(\beta_m x)\}$$

$$- \left[\left\{\left(\frac{k\pi}{\beta_m l}\right)^2 + 1\right\}\cosh(\beta_m l)\cos(\beta_m x) - \left\{\left(\frac{k\pi}{\beta_m l}\right)^2 - 1\right\}\cos(\beta_m l)\cosh(\beta_m x)\right]$$

$$\times \{\sin(\beta_m l)\cosh(\beta_m l) - \cos(\beta_m l)\sinh(\beta_m l)\}$$

$$- 2\sin(\beta_m l)\sinh(\beta_m l)\{\sin(\beta_m l)\cosh(\beta_m x) + \sinh(\beta_m l)\cos(\beta_m x)\}, \tag{B7}$$

$$W_m^{(2)} = \sin(\beta_m l)\sinh(\beta_m l)\{\cos(\beta_m l)\cosh(\beta_m x) - \cosh(\beta_m l)\cos(\beta_m x)\}$$

$$-\cos(\beta_m l)\cosh(\beta_m l)\{\sin(\beta_m l)\sinh(\beta_m x) - \sinh(\beta_m l)\sin(\beta_m x)\}. \tag{B8}$$